HADAMARD GAP SERIES IN GROWTH SPACES

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Abstract. Let h_v^∞ be the class of harmonic functions in the unit disk which admit a two-sided radial majorant $v(r)$. We consider functions v that fulfill a doubling condition. We characterize functions in h_v^∞ that are represented by Hadamard gap series in terms of their coefficients, and as a corollary we obtain a characterization of Hadamard gap series in Bloch-type spaces for weights with a doubling property. We show that if $u \in h_v^\infty$ is represented by a Hadamard gap series, then u will grow slower than v or oscillate along almost all radii. We use the law of the iterated logarithm for trigonometric series to find an upper bound on the growth of a weighted average of the function u, and we show that the estimate is sharp.

1. Introduction

1.1. **Growth spaces of harmonic functions.** Let v be a positive increasing continuous function on $[0, 1)$, assume that $v(0) = 1$ and $\lim_{r\to 1} v(r) = +\infty$. We study the growth spaces of harmonic functions in the unit disk:

$$h_v^\infty = \{u : \mathbf{D} \to \mathbf{R}, \Delta u = 0, |u(z)| \le K v(|z|) \text{ for some } K > 0\}. \tag{1}$$

In this article we mainly deal with weights which satisfy the doubling condition

$$v(1 - d) \le D v(1 - 2d). \tag{2}$$

For such weights we characterize functions in h_v^∞ represented by Hadamard gap series. Further, using this characterization, we study radial behavior of such functions in more detail.

Growth spaces appear naturally as duals to the classical spaces of harmonic and analytic functions; a similar class of harmonic functions satisfying only one-sided estimate with $v(r) = \log \frac{1}{1-r}$ was introduced by B. Korenblum in [8]. General growth spaces can be found in the works of L. Rubel and A. Shields, and A. Shields and D. Williams, see [11, 13]. Multidimensional analogs were recently considered in [1, 6]. Various results on coefficients of functions in growth spaces were obtained in [2].

1.2. **Hadamard gap series.** Hadamard gap series are functions of the form

$$f(z) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k}, \quad a_{n_k} \in \mathbf{C},$$

where $n_{k+1} > \lambda n_k$ and $\lambda > 1$; for the harmonic case we take $u = \Re f$. For various spaces of analytic and harmonic functions Hadamard gap series in these spaces can be characterized in terms of the coefficients a_n. For example, α-Bloch spaces are

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studied in [18], in [14] more general Bloch-type spaces are considered, for Q_K spaces results are obtained in [17], and the case of growth spaces of harmonic functions with normal weights (i.e. weights that essentially grow between two powers $(1-r)^{-a}$ and $(1-r)^{-b}$) is recently treated in [19]. All these results give necessary and sufficient conditions on the coefficients of the series in the form $|a_n| \leq f(n)$ for an appropriate function f. For general growth spaces, in particular for those with slow growing function v, such one-term characterization does not exist. We prove that the condition

$$\sum_{n_k \leq N} |a_{n_k}| \leq Cv(1-1/N)$$

characterizes all Hadamard gap series in h_v^∞ for any v that satisfies (2). This generalizes previous results for $v = \log \frac{1}{1-r}$ [9] and also one for normal weights [19]. The proof is based on a beautiful theorem of J.-P. Kahane, M.Weiss and G. Weiss [7]. As a corollary we obtain the characterization of Hadamard gap series in Bloch-type spaces with weights satisfying a doubling condition.

1.3. **Radial oscillation.** Boundary behavior of harmonic functions in growth spaces was studied in [4, 9, 6]. It turns out that a harmonic function $u \in h_v^\infty$ cannot grow as fast as v along most of the radii, precise statements can be found in [4, 6]. In order to describe typical radial behavior of functions in h_v^∞ for the Korenblum space with $v = \log \frac{1}{1-r}$ the following weighted average was introduced in [9]

$$I_u(R,\phi) = \int_{1/2}^{R} \frac{u(re^{i\phi})}{(1-r)\left(\log \frac{1}{1-r}\right)^2} dr, \quad R \in (0,1), \quad \phi \in (-\pi, \pi].$$

The law of the iterated logarithm applied to this average gives an upper estimate for $I_u(R,\phi)$ and shows that in general it grows slower than the similar average of $|u(re^{i\phi})|$. This result is referred to as radial oscillation of harmonic functions in h_v^∞, see [9] for details.

One of the aims of this article is to find an appropriate weighted average for other growth spaces h_v^∞. We note that the argument in [9] cannot be applied for fast growing weights, though it seems plausible that the result still holds. We obtain radial oscillation for the case when $u \in h_v^\infty$ is represented by a Hadamard gap series, the only restriction on v is the doubling condition (2). The precise statement is the following: Let $u \in h_v^\infty$ be a Hadamard gap series, define

$$I_u^{(v)}(R,\phi) = \int_{1/2}^{R} \frac{u(re^{i\phi})dv(r)}{v^2(r)}.$$

Then for almost every ϕ

$$\lim_{R\to 1} \frac{I_u^{(v)}(R,\phi)}{\sqrt{\log v(R) \log\log\log v(R)}} \leq C.$$

The proof uses the law of the iterated logarithm for trigonometric series from [16], see also [12]. A straightforward estimate gives $|I_u^{(v)}(R,\phi)| \leq K \log v(R)$; our result shows that the growth is much slower for almost every ϕ. We will also give an example of a function $u \in h_v^\infty$ such that

$$\limsup \frac{I_{|u|}^{(v)}(R,\phi)}{\log v(R)} > 0$$

for almost every ϕ. This difference in the asymptotic behavior of $I_u^{(v)}$ and $I_{|u|}^{(v)}$ indicates oscillation.

1.4. **One-sided estimates.** We consider also classes of harmonic functions satisfying one-sided estimate only,

$$k_v^\infty = \{u : \mathbf{D} \to \mathbf{R}, \Delta u = 0, u(z) \leq Kv(|z|) \text{ for some } K > 0\}. \tag{3}$$

In general there is a substantial difference between classes h_v^∞ and k_v^∞, however it cannot be observed on functions represented by Hadamard gap series.

1.5. **Organization of the paper.** In section 2 we first prove an estimate on the coefficients of functions in k_v^∞ for general v. For v that fulfills the doubling condition we show that if u is a Hadamard gap series then $u \in h_v^\infty$ if and only if $u \in k_v^\infty$ and characterize such series in terms of their coefficients. As a corollary we get a characterization of Hadamard gap series in general Bloch spaces. In section 3 we prove the result on radial oscillation and show that it is precise.

For convenience we define a new function $g : [1, \infty) \to [1, \infty)$ such that $g(x) = v(1 - x^{-1})$. Then (2) is equivalent to

$$g(2x) \leq Dg(x). \tag{4}$$

We will keep this notation throughout the paper, and constants K in (1), D and λ will preserve their identities. Other constants need not be the same in each place. From now on we will write $I_u(R, \phi)$ instead of $I_u^{(v)}(R, \phi)$ for simplicity.

2. Coefficient estimates and Hadamard gap series

2.1. **Coefficient estimates.** Each real harmonic function u in the unit disk can be written as

$$u(re^{i\phi}) = \Re \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathbf{C},$$

where

$$a_n = \frac{r^{-n}}{\pi} \int_{-\pi}^{\pi} u(re^{i\phi}) e^{-in\phi} d\phi, \qquad n \geq 0.$$

We will now estimate the size of $|a_n|$ for $u \in k_v^\infty$.

Proposition 2.1. *Assume that $u \in k_v^\infty$ and*

$$u(re^{i\phi}) = \Re \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathbf{C}.$$

Then $|a_n| \leq Cg(n)$ for some $C = C(K)$, where K is as in (3).

This proposition is a generalization of Theorem 1 (i) in [8, p. 209], where $v(r) = \log \frac{1}{1-r}$. It also generalizes Theorem 1.12 (a) in [2], which is valid for $u \in h_v^\infty$ if v satisfies (2).

Proof. We have

$$|a_n| = \left| \frac{r^{-n}}{\pi} \int_{-\pi}^{\pi} u(re^{i\phi}) e^{-in\phi} d\phi \right| \leq \frac{r^{-n}}{\pi} \int_{-\pi}^{\pi} |u(re^{i\phi})| d\phi.$$

Note that

$$\int_{-\pi}^{\pi} |u(re^{i\phi})| d\phi = \int_{u(re^{i\phi}) \geq 0} u(re^{i\phi}) d\phi - \int_{u(re^{i\phi}) < 0} u(re^{i\phi}) d\phi =$$

$$2\int_{u(re^{i\phi})\geq 0} u(re^{i\phi})d\phi - \int_{-\pi}^{\pi} u(re^{i\phi})d\phi \leq 4\pi K v(r) - 2\pi u(0).$$

This implies that for any $r \in (0,1)$

$$|a_n| \leq C_1 r^{-n} v(r) = C_1 r^{-n} g\left(\frac{1}{1-r}\right),$$

where C_1 depends on K. Let $r = 1 - 1/n$, then $|a_n| \leq Cg(n)$. □

The estimate $|a_n| \leq Cg(n)$ is in general not enough to imply that a function is in k_v^∞, but in some cases it is, we will come back to that in Corollary 2.5.

2.2. **Characterization of Hadamard gap series in growth spaces.** We will prove the following:

Theorem 2.2. *Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers such that $n_{k+1} \geq \lambda n_k$ for each k, where $\lambda > 1$. Assume v satisfies (2) and let*

$$u(z) = \Re \sum_k a_{n_k} z^{n_k}, \qquad a_{n_k} \in \mathbf{C},$$

where the series converges in the unit disk. Then $u \in k_v^\infty$ if and only if there exists γ such that $\sum_{n_k \leq N} |a_{n_k}| \leq \gamma\, g(N)$ for any $N \in \mathbf{N}$.

This theorem generalizes Proposition 5.1 in [9], which is based on results from [8] and [7].

Proof. We first assume that $u \in k_v^\infty$. Let $r_N = 2^{-1/N}$ and let M be a constant. We have

$$u(r_N e^{i\phi}) = \Re \sum_{n_k \leq NM} a_{n_k} r_N^{n_k} e^{in_k\phi} + \Re \sum_{n_k > NM} a_{n_k} r_N^{n_k} e^{in_k\phi} = s_N(\phi) + t_N(\phi).$$

By Proposition 2.1

$$|t_N(\phi)| \leq C \sum_{n_k > NM} g(n_k) 2^{-n_k/N}.$$

Since u is a Hadamard gap series there exists $\lambda > 1$ such that $n_{j+1} > \lambda n_j$ for all j. We may assume that $\lambda \leq 2$, and by adding extra terms with $a_j = 0$ we may also assume that $n_{j+1} \leq 4n_j$. Let $n_k \geq NM$. Then by (4)

$$\frac{g(n_{k+1})2^{-n_{k+1}/N}}{g(n_k)2^{-n_k/N}} \leq \frac{g(4n_k)}{g(n_k)} 2^{-n_k(\lambda-1)/N} \leq D^2 2^{-M(\lambda-1)} < q < 1 \tag{5}$$

when M is sufficiently large. Using this and (4), the reminder term can then be estimated as follows

$$|t_N(\phi)| \leq C \sum_{n_k > NM} g(n_k) 2^{-n_k/N} \leq C_q g(NM) \leq Bg(N).$$

By Theorem I in [7] there exists $\alpha = \alpha(\lambda) > 0$ and $\phi_0 \in (-\pi, \pi)$ such that

$$s_N(\phi_0) \geq \alpha \sum_{n_k \leq NM} |a_{n_k} r_N^{n_k}|.$$

We have $r_N \leq 1 - \frac{1}{3N}$, so

$$g\left(\frac{1}{1-r_N}\right) \leq g(3N) \leq D^2 g(N).$$

Thus

$$\begin{aligned}\sum_{n_k\le N}|a_{n_k}| &\le 2\sum_{n_k\le NM}|a_{n_k}r_N^{n_k}| \le 2\alpha^{-1}\left(u(r_Ne^{i\phi_0})+|t_N(\phi_0)|\right)\\ &\le 2\alpha^{-1}\left(g\left(\frac{1}{1-r_N}\right)+Bg(N)\right)\le \gamma g(N).\end{aligned}$$

Now assume there exists γ such that $\sum_{n_k\le N}|a_{n_k}|\le \gamma\, g(N)$ for any $N\in\mathbf{N}$. Let $r_N<r\le r_{N+1}$. Then $\frac{N}{\log 2}\le\frac{1}{1-r}$, and using (4) and (5) we get

$$\begin{aligned}u(re^{i\phi}) &\le \sum_{n_k\le NM}|a_k|+\sum_{n_k>NM}\gamma g(n_k)r^{n_k}\le C_1g(MN)\\ &\le C_2g\left(\frac{N}{\log 2}\right)\le C_2g\left(\frac{1}{1-r}\right).\end{aligned}\tag{6}$$

□

Corollary 2.3. *If u is a Hadamard gap series and v fulfills* (2)*, then $u\in k_v^\infty \Leftrightarrow u\in h_v^\infty$.*

Proof. If $\sum_{n_k\le N}|a_{n_k}|\le\gamma\, g(N)$ is true for u, then of course it is also true for $-u$. □

Corollary 2.4. *Let f be a holomorphic Hadamard gap series in the unit disk and let v satisfy* (2)*. If $\Re f\in k_v^\infty$, then $\Re f,\Im f\in h_v^\infty$.*

For another result of this type, see [5]. There it is proven that if $v(r)=\left(\frac{1}{1-r}\right)^\alpha$ for $\alpha>1$ and f is holomorphic, then $\Re f\in k_v^\infty$ implies $\Re f\in h_v^\infty$ and $\Im f\in h_v^\infty$. For some v the fact that $u\in k_v^\infty$ will imply that $u\in h_v^\infty$, see [10] and [3].

Corollary 2.5. *If u is a Hadamard gap series and g is such that for each $q>1$ there is $A(q)>1$ that satisfies*

$$x>qy \Rightarrow g(x)>Ag(y)\tag{7}$$

when $y>y_0$, then $|a_{n_k}|\le Cg(n_k)$ implies $u\in h_v^\infty$.

Proof. We get

$$\sum_{n_k\le N}|a_{n_k}|\le C\sum_{n_k\le N}g(n_k)\le C_1g(N)$$

whenever $\frac{n_{k+1}}{n_k}\ge\lambda>1$, where C_1 depends on A. □

For example when $g(x)=x^\gamma,\gamma>0$, (7) is satisfied. This corollary is slightly more general than Theorem 2.3 in [19], which concerns normal weights.

For general v, the inequality $|a_n|\le Cg(n)$ does not imply that $u\in k_v^\infty$ even when u is represented by a Hadamard gap series. It is not difficult to construct g (for example $g(x)=\log x$) such that for any C

$$\sum_{k\le n}g(2^k)>Cg(2^n)\qquad\text{for some } n=n(C).$$

Then by the previous theorem $u(z)=\sum_{k=1}^\infty g(2^k)z^{2^k}\notin k_v^\infty$.

2.3. **Application to Bloch-type functions.** Let μ be a decreasing continuous function on $[0,1)$ and assume $\lim_{r\to 1^-}\mu(r)=0$. A harmonic function is in the Bloch-type space $\mathcal{B}_\mu$ if

$$||u||_{\mathcal{B}_\mu}=\sup_{z\in\mathbf{D}}(|u(0)|+\mu(|z|)|\nabla u(z)|)<\infty.$$

Hadamard gap series in Bloch-type spaces have been characterized in terms of their coefficients for many concrete μ. For $\mu(r)=(1-r^2)^\alpha$, a Hadamard gap series is in $\mathcal{B}_\mu$ if and only if $\limsup_{j\to\infty}|a_j|n_j^{1-\alpha}<\infty$, this is from [18]. In [14] the condition

$$\limsup_{j\to\infty} n_j|a_j|\mu(1-1/n_j)<\infty$$

is obtained under some restrictions on the weight, in particular for the weight $\mu(r)=(1-r)\Pi_{j=1}^k\log^{[j]}\frac{e^k}{1-r}$, where $\log^{[j]}$ is the logarithm applied j times. Using Theorem 2.2 we will obtain a more general result, where we only assume that μ does not decay too fast, more presicely, that it fullfills a condition similar to (2):

$$\mu\left(1-\frac{d}{2}\right)\geq B\mu(1-d). \tag{8}$$

Corollary 2.6. *Let μ fulfill* (8)*. A Hadamard gap series is in the Bloch-type space $\mathcal{B}_\mu$ if and only if there is a C such that*

$$\sum_{n_k\leq N} n_k|a_{n_k}|\leq C\,\frac{1}{\mu(1-1/N)}.$$

Proof. A function $u(z)=\Re\sum_k a_{n_k}z^{n_k}$ is in $\mathcal{B}_\mu$ if and only if $\Re\sum_k n_k a_{n_k}z^{n_k}$ is in h_v^∞ for $v(r)=1/\mu(r)$. Then the result follows from Theorem 2.2. □

Corollary 2.7. *Let μ fulfill* (8)*. If u is a Hadamard gap series and μ is such that for each $q>1$ there is $A(q)>1$ that satisfies*

$$x>qy\ \Rightarrow\ \mu(1-1/y)>A\mu(1-1/x) \tag{9}$$

when $y>y_0$, then $\sup_k n_k|a_{n_k}|\mu(1-1/n_k)\leq C$ implies $u\in\mathcal{B}_\mu$.

Proof. This follows from Corollary 2.5 and 2.6. □

2.4. **Examples.** We give examples of functions in h_v^∞ when v fulfills the doubling condition (2). These examples will be used later. Let $A>1$, $b_0=1$ and define b_n by induction as

$$b_{n+1}=\min\{l\in\mathbf{N}:g(2^l)>Ag(2^{b_n})\}. \tag{10}$$

Lemma 2.8. *Assume v satisfies (2). Let*

$$u(z)=\Re\sum_{k=0}^\infty g(2^{b_k})z^{2^{b_k}},\qquad z\in\mathbf{D},$$

where b_n is given as above. Then $u\in h_v^\infty$.

Proof. We have

$$\sum_{n=0}^N g(2^{b_n})\leq g(2^{b_N})\sum_{n=0}^N\frac{1}{A^n}\leq\frac{A}{A-1}\,g(2^{b_N})$$

Then by (iii) of the previous theorem $u\in h_v^\infty$. □

When $g(t) = t^a$ or $g(t) = (\log t)^a$, examples of function in h_v^∞ are respectively

$$u(z) = \Re \sum_{k=0}^{\infty} 2^{ka} z^{2^k} \quad \text{and} \quad u(z) = \Re \sum_{k=0}^{\infty} 2^{ka} z^{2^{2^k}}.$$

3. Oscillation

3.1. **A preliminary estimate.** We will need the following lemma to prove our main result. By the relation $a \lesssim b$ we mean that there exists a constant C that only depends on D, K and λ such that $a \le Cb$. If $a \lesssim b$ and $a \gtrsim b$, we write $a \simeq b$. The function v is as in the introduction.

Lemma 3.1. *Let v satisfy* (2). *There exist C and n_0, which depend only on D, such that*

$$\int_0^s \frac{r^{n-1}}{v(r)} dr \le C \frac{s^n}{nv(s)}$$

for $n \ge n_0$ and $s \le 1 - \frac{1}{n}$.

When we use $\lesssim$ in the proof of this lemma, the constants will only depend on D.

Proof. Let l be such that

$$1 + 2\log(2D^l) \le 2^l, \tag{11}$$

so l only depends on D. Let $\rho_0 = 0$ and choose ρ_j such that $v(\rho_j) = \gamma v(\rho_{j-1})$ where $\gamma = D^l$. Choose k such that

$$\rho_k \le s \le \rho_{k+1}. \tag{12}$$

Since $v(r) \ge \gamma^m$ when $r \ge \rho_m$, we have

$$\int_0^s \frac{r^{n-1}}{v(r)} dr \le \sum_{m=0}^{k-1} \frac{1}{\gamma^m} \int_{\rho_m}^{\rho_{m+1}} r^{n-1} dr + \frac{1}{\gamma^k} \int_{\rho_k}^{s} r^{n-1} dr \le \frac{1}{n} \left(\sum_{m=0}^{k-1} \frac{\rho_{m+1}^n}{\gamma^m} + \frac{s^n}{\gamma^k} \right). \tag{13}$$

We will now show that

$$\sum_{m=0}^{k-1} \frac{\rho_{m+1}^n}{\gamma^m} \lesssim \frac{\rho_k^n}{\gamma^{k-1}} \lesssim \frac{s^n}{\gamma^k}. \tag{14}$$

This is true if

$$2\frac{\rho_m^n}{\gamma^{m-1}} \le \frac{\rho_{m+1}^n}{\gamma^m}$$

for any $1 \le m \le k-1$. That is equivalent to $(2\gamma)^{1/n}\rho_m \le \rho_{m+1}$. Assume $s \le 1 - \frac{1}{n}$, then by (12) we have $1/n \le 1 - \rho_k$, and for $n \ge n_0$ for some n_0 large enough,

$$(2\gamma)^{1/n} = e^{\log(2\gamma)/n} \le 1 + 2\log(2\gamma)/n \le 1 + 2\log(2\gamma)(1 - \rho_k).$$

We need

$$(1 + 2\log(2\gamma)(1 - \rho_k))\rho_{m-1} \le \rho_m$$

for any k and $m \le k$. It is enough to check that

$$(1 + 2\log(2\gamma)(1 - \rho_k))\rho_{k-1} \le \rho_k.$$

(This can be seen by fixing m and letting k vary.) We rewrite this as

$$\rho_k \ge \frac{(1 + 2\log(2\gamma))\rho_{k-1}}{1 + 2\log(2\gamma)\rho_{k-1}} = 1 - \frac{1 - \rho_{k-1}}{1 + 2\log(2\gamma)\rho_{k-1}}. \tag{15}$$

We have

$$v(\rho_k) = \gamma v(\rho_{k-1}) = \gamma v(1-(1-\rho_{k-1})) \tag{16}$$

and by (2) and (11),

$$v\left(1 - \frac{1-\rho_{k-1}}{1+2\log(2\gamma)\rho_{k-1}}\right) \le \gamma v(1-(1-\rho_{k-1})). \tag{17}$$

Then (16) and (17) give (15) since v is increasing, and therefore (14) is true. Furthermore,

$$\frac{s^n}{\gamma^k} \lesssim \frac{s^n}{v(s)}. \tag{18}$$

Then by (13), (14) and (18), we get

$$\int_0^s \frac{r^{n-1}}{v(r)} dr \lesssim \frac{s^n}{nv(s)}$$

when $n \ge n_0$ and $s \le 1 - \frac{1}{n}$. □

3.2. **Main result.** The following theorem by Mary Weiss will be used to prove our next theorem:

Theorem A. *Let*

$$S(x) = \sum_{k=0}^{\infty} a_j \cos(n_j\phi) + \sum_{k=0}^{\infty} b_j \sin(n_j\phi)$$

where $n_{j+1}/n_j > \lambda > 1$ for all j. We write

$$B_N = \left(\frac{1}{2}\sum_{j=0}^{N}(a_j^2+b_j^2)\right)^{1/2}, \qquad M_N = \max_{1\le j\le N}(a_j^2+b_j^2)^{1/2},$$

$$S_N(\phi) = \sum_{j=0}^{N} a_j \cos n_j\phi + \sum_{j=0}^{N} b_j \sin(n_j\phi).$$

If $B_N \to \infty$ and $M_N = o(B_N/(\log\log B_N)^{1/2})$ as $N \to \infty$, then

$$\limsup_{N\to\infty} \frac{S_N(\phi)}{(2B_N^2 \log\log B_N)^{1/2}} = 1 \qquad \text{for a.e. } \phi.$$

For $u \in k_v^\infty$ we define the weighted average

$$I_u(R,\phi) = \int_{1/2}^{R} \frac{u(re^{i\phi})dv(r)}{(v(r))^2}.$$

See for example [15] for the definition of a Riemann-Stieltjes integral. It follows from (3) that

$$|I_u(R,\phi)| \le K \log v(R).$$

We will prove the following:

Theorem 3.2. *If v satisfies (2) and $u \in h_v^\infty$ is represented by a Hadamard gap series, then*

$$\limsup_{R\to 1} \frac{I_u(R,\phi)}{(\log v(R) \log\log\log v(R))^{1/2}} \le C \tag{19}$$

for almost all ϕ, where C depends on D, K and λ.

Proof. Let

$$u(re^{i\phi}) = \sum_{j=0}^{\infty} \alpha_j r^{n_j} \cos(n_j\phi) + \sum_{j=0}^{\infty} \beta_j r^{n_j} \sin(n_j\phi)$$

where $\alpha_j, \beta_j \in \mathbf{R}$, then

$$I_u(R,\phi) = \sum_{j=0}^{\infty} \alpha_j \int_{1/2}^{R} \frac{r^{n_j}\,dv(r)}{(v(r))^2} \cos n_j\phi + \sum_{j=0}^{\infty} \beta_j \int_{1/2}^{R} \frac{r^{n_j}\,dv(r)}{(v(r))^2} \sin n_j\phi.$$

For simplicity we will only prove the result for the sum with cosines since the proof with sines is the same, so from now on we let $u(re^{i\phi}) = \sum_{k=0}^{\infty} \alpha_j r^{n_j} \cos(n_j\phi)$ and

$$I_u(R,\phi) = \sum_{j=0}^{\infty} \alpha_j \int_{1/2}^{R} \frac{r^{n_j}\,dv(r)}{(v(r))^2} \cos n_j\phi.$$

Since u is a Hadamard gap series there exists $\lambda > 1$ such that $n_{j+1} > \lambda n_j$ for all j. As before we may assume that $\lambda \le 2$ and $n_{j+1} \le 4n_j$. Let

$$c_j = \int_{1/2}^{1} \frac{r^{n_j}\,dv(r)}{(v(r))^2}$$

and

$$S_N(\phi) = \sum_{j=0}^{N} \alpha_j c_j \cos n_j\phi.$$

Let also $r_N = 1 - \frac{1}{n_N}$ and suppose $R \in [r_N, r_{N+1})$. We will first show that

$$|I_u(R,\phi) - S_N(\phi)| \lesssim 1. \tag{20}$$

By (3) and (2) we have

$$|I_u(R,\phi) - I_u(r_N,\phi)| \le K \int_{r_N}^{r_{N+1}} \frac{dv(r)}{v(r)} = K \log \frac{v(r_{N+1})}{v(r_N)} \lesssim 1. \tag{21}$$

Moreover,

$$I_u(r_N,\phi) \quad = \quad \sum_{j=0}^{\infty} \alpha_j \int_{1/2}^{r_N} \frac{r^{n_j}\,dv(r)}{(v(r))^2} \cos n_j\phi = \sum_{j=0}^{\infty} \alpha_j b_{j,N} \cos n_j\phi$$

and

$$b_{j,N} \le \int_0^{r_N} \frac{r^{n_j}\,dv(r)}{(v(r))^2} = \left[\frac{-r^{n_j}}{v(r)}\right]_0^{r_N} + n_j \int_0^{r_N} \frac{r^{n_j-1}}{v(r)} dr \le n_j \int_0^{r_N} \frac{r^{n_j-1}}{v(r)} dr.$$

By Lemma 3.1,

$$b_{j,N} \lesssim \frac{r_N^{n_j}}{g(n_N)}$$

when $n_j \ge \max\{n_N, n_0\}$. Let M be a constant such that $D^2 e^{-M(\lambda-1)} < q < 1$. By Proposition 2.1,

$$\sum_{n_j > n_N M} |\alpha_j b_{j,N}| \lesssim \sum_{n_j > n_N M} g(n_j) \frac{r_N^{n_j}}{g(n_N)} \le \sum_{n_j > n_N M} \frac{g(n_j)}{g(n_N)} e^{-n_j/n_N}. \tag{22}$$

For $n_j > n_N M$ we have

$$\frac{g(n_{j+1})e^{-n_{j+1}/n_N}}{g(n_j)e^{-n_j/n_N}} \le \frac{g(4n_j)}{g(n_j)} e^{-n_j(\lambda-1)/n_N} \le D^2 e^{-M(\lambda-1)} < q < 1,$$

thus (22) is bounded by a constant independent of N. Furthermore, by Theorem 2.2,

(23)

$$\sum_{n_N<n_j\le n_N M}|\alpha_j b_{j,N}|\lesssim \sum_{n_N<n_j\le n_N M}\frac{|\alpha_j|r_N^{n_j}}{g(n_N)}\le\frac{1}{g(n_N)}\sum_{n_j\le n_N M}|\alpha_j|\lesssim\frac{g(n_N M)}{g(n_N)}\lesssim 1.$$

We now need to estimate $\sum_{n_j\le n_N}|\alpha_j(b_{j,N}-c_j)|$. For $j\le N$,

$$|b_{j,N}-c_j|=\int_{r_N}^1\frac{r^{n_j}\,dv(r)}{(v(r))^2}\le\int_{r_N}^1\frac{dv(r)}{(v(r))^2}=\frac{1}{v(r_N)},$$

then by Theorem 2.2 and (4)

$$\sum_{n_j\le n_N}|\alpha_j||b_{j,N}-c_j|\le\frac{1}{v(r_N)}\sum_{n_j\le n_N}|\alpha_j|\lesssim\frac{g(n_N)}{g(n_N)}=1. \tag{24}$$

Thus (20) follows from (21), (22) , (23) and (24):

$$\begin{aligned}&|I_u(R,\phi)-S_N(\phi)|\le|I_u(R,\phi)-I_u(r_N,\phi)|+|I_u(r_N,\phi)-S_N(\phi)|\\ \lesssim 1+&\sum_{n_j\le n_N}|\alpha_j||b_{j,N}-c_j|+\sum_{n_N<n_j\le n_N M}|\alpha_j b_{j,N}|+\sum_{n_j>n_N M}|\alpha_j b_{j,N}|\lesssim 1.\end{aligned}$$

We claim that

$$\sum_{j=0}^N(\alpha_j c_j)^2\lesssim\log g(n_N) \tag{25}$$

for any N. Let $q_0=1$ and choose q_j such that $g(q_j)=2g(q_{j-1})$. When $n_N\in(q_p,q_{p+1}]$ we have $\log g(n_N)\lesssim p$, so (25) would follow if

$$\sum_{q_k<n_j\le q_{k+1}}(\alpha_j c_j)^2\lesssim 1 \tag{26}$$

for all k. We have

$$c_j\le\int_0^1\frac{r^{n_j}\,dv(r)}{(v(r))^2}=\left[\frac{-r^{n_j}}{v(r)}\right]_0^1+n_j\int_0^1\frac{r^{n_j-1}}{v(r)}dr=n_j\int_0^1\frac{r^{n_j-1}}{v(r)}dr \tag{27}$$

and

$$\int_{1-1/n_j}^1\frac{r^{n_j-1}}{v(r)}dr\le\frac{1}{v(1-1/n_j)}\int_{1-1/n_j}^1 r^{n_j-1}dr\le\frac{1}{n_j g(n_j)}. \tag{28}$$

By Lemma 3.1,

$$\int_0^{1-1/n_j}\frac{r^{n_j-1}}{v(r)}dr\lesssim\frac{1}{n_j g(n_j)}$$

for $n_j\ge n_{j_0}$, and by this, (27) and (28) we get

$$c_j\lesssim\frac{1}{g(n_j)}. \tag{29}$$

Thus for $q_k<n_j\le q_{k+1}$, we have $c_j\lesssim 1/g(q_k)$. Then by Theorem 2.2,

$$\sum_{q_k<n_j\le q_{k+1}}(\alpha_j c_j)^2\le\frac{1}{(g(q_k))^2}\left(\sum_{n_j\le q_{k+1}}|\alpha_j|\right)^2\lesssim\frac{(g(q_{k+1}))^2}{(g(q_k))^2}\lesssim 1$$

and (26) is proved.

Now let

$$B_N = \left(\frac{1}{2} \sum_{j=0}^{N} (\alpha_j c_j)^2 \right)^{1/2}$$

and

$$M_N = \max_{1 \leq j \leq N} |\alpha_j c_j|.$$

We want to use Theorem 1 in [16] to show that

$$\limsup_{N \to \infty} \frac{S_N(\phi)}{(\log g(n_N) \log \log \log g(n_N))^{1/2}} \lesssim 1$$

for almost all ϕ. Then (19) will follow from (20).

If B_N is bounded, then S_N is bounded a.e. (see for example Theorem 6.3 in [20, p. 203]), so there is nothing to prove. By Proposition 2.1 and (29) we have $M_N \lesssim 1$, so if $B_N \to \infty$, the conditions of Theorem A are fulfilled. Since $B_N \lesssim (\log g(n_N))^{1/2}$ by (25), we get

$$\limsup_{N \to \infty} \frac{S_N(\phi)}{(\log g(n_N) \log \log \log g(n_N))^{1/2}} \lesssim \limsup_{N \to \infty} \frac{S_N(\phi)}{(2B_N^2 \log \log B_N)^{1/2}} = 1$$

for almost all ϕ, and we are done.

□

Remark. The same result is in general not true if u is replaced by $|u|$, as the next lemma shows. This indicates that the function is oscillating.

Lemma 3.3. *Let v satisfy (2). There is a Hadamard gap series $u \in h_v^\infty$ such that*

$$\limsup_{R \to 1^-} \frac{I_{|u|}(R, \phi)}{\log v(R)} > 0 \tag{30}$$

for all ϕ in a set of positive measure.

Proof. Let $u(z) = \Re \sum_{k=0}^{\infty} g(2^{b_k}) z^{2^{b_k}}$, where b_k are defined by (10). By (3) it follows that $I_{|u|}(R, \phi) \leq K \log v(R)$. Assume we have also shown that

$$||I_{|u|}(R, \cdot)||_1 \geq c \log v(R) \tag{31}$$

for all $R \geq R_0 > 1/2$. Let

$$E_R = \{\phi \in (-\pi, \pi] : I_{|u|}(R, \phi) > \frac{c}{2} \log v(R)\},$$

then

$$\begin{aligned} ||I_{|u|}(R, \cdot)||_1 &= \int_{E_R} I_{|u|}(R, \phi) d\phi + \int_{E_R^c} I_{|u|}(R, \phi) d\phi \\ &\leq |E_R| K \log v(R) + (1 - |E_R|) \frac{c}{2} \log v(R), \end{aligned}$$

hence $|E_R| \geq c/(2K - c)$. Then there exists a set of positive measure such that (30) is fulfilled.

Now it remains to show (31). Fix r and choose N such that $1 - 1/2^{b_N} \leq r \leq 1 - 1/2^{b_{N+1}}$. By using that $g(2^{b_{k+1}}) \leq ADg(2^{b_k})$, we get

$$\int_0^{2\pi} |u(re^{i\phi})|^2 d\phi = \sum_{k=0}^{\infty} g(2^{b_k})^2 r^{2^{b_k+1}} \geq g(2^{b_N})^2 \left(1 - \frac{1}{2^{b_N}}\right)^{2^{b_N+1}} \geq \frac{1}{16A^2D^2} (v(r))^2.$$

Furthermore,

$$\int_0^{2\pi} |u(re^{i\phi})|^2 d\phi \leq \max_\phi |u(re^{i\phi})| \int_0^{2\pi} |u(re^{i\phi})| d\phi \leq Kv(r) \int_0^{2\pi} |u(re^{i\phi})| d\phi,$$

hence

$$\int_0^{2\pi} |u(re^{i\phi})| d\phi \geq \frac{1}{16A^2D^2K} v(r).$$

Then

$$\int_0^{2\pi} I_{|u|}(R,\phi) d\phi = \int_{1/2}^R \int_0^{2\pi} |u(re^{i\phi})| d\phi \frac{dv(r)}{(v(r))^2} \geq c \log v(R)$$

for all $R \geq R_0 > 1/2$, and we are done. □

3.3. **Sharpness of Theorem 3.2.** The estimate is precise: There exists a function in h_v^∞ such that for some $a > 0$

$$\limsup_{R\to 1} \frac{I_u(R,\phi)}{(\log v(R) \log\log\log v(R))^{1/2}} \geq a \tag{32}$$

for almost all $\phi \in (-\pi, \pi]$.

To see this, let u be the function from Lemma 2.8 with $A = 2$. Then

$$I_u(R,\phi) = \sum_{j=0}^\infty g(2^{b_j}) \int_{1/2}^R \frac{r^{2^{b_j}} dv(r)}{(v(r))^2} \cos 2^{b_j}\phi.$$

Let $r_j = 1 - \frac{1}{2^{b_j}}$. It can be shown that as in the proof of theorem 3.2,

$$|I_u(R,\phi) - S_N(\phi)| \lesssim 1$$

when $R \in (r_N, r_{N+1})$. We also have

$$c_j \geq r_j^{2^{b_j}} \int_{r_j}^1 \frac{dv(r)}{(v(r))^2} \gtrsim \frac{1}{v(r_j)},$$

so $c_j \simeq \frac{1}{v(r_j)}$. By this and (10),

$$B_N^2 = \frac{1}{2} \sum_{j=0}^N (g(2^{b_j})c_j)^2 \simeq N \simeq \log g(2^{b_N}) = \log v(r_N).$$

Then

$$\limsup_{R\to 1} \frac{I_u(R,\phi)}{(\log v(R) \log\log\log v(R))^{1/2}} \gtrsim \limsup_{N\to\infty} \frac{S_N(\phi)}{(B_N^2 \log\log B_N)^{1/2}} = 1$$

for almost all ϕ by Theorem A.

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